A REMARK ON 'SOME NUMERICAL RESULTS IN COMPLEX DIFFERENTIAL GEOMETRY'

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ABSTRACT. In this note we verify certain statement about the operator Q_K constructed by Donaldson in [3] by using the full asymptotic expansion of Bergman kernel obtained in [2] and [4].

In order to find explicit numerical approximation of Kähler-Einstein metric of projective manifolds, Donaldson introduced in [3] various operators with good properties to approximate classical operators. See the discussions in Section 4.2 of [3] for more details related to our discussion. In this note we verify certain statement of Donaldson about the operator Q_K in Section 4.2 by using the full asymptotic expansion of Bergman kernel derived in [2, Theorem 4.18] and [4, §3.4]. Such statement is needed for the convergence of the approximation procedure.

Let (X, ω, J) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$, and let (L, h^L) be a holomorphic Hermitian line bundle on X. Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature R^L. We assume that

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{1}$$

Let $g^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω, J. Let dv_X be the Riemannian volume form of (TX, g^{TX}), then $dv_X = \omega^n/n!$. Let $d\nu$ be any volume form on X. Let η be the positive function on X defined by

$$dv_X = \eta \, d\nu. \tag{2}$$

The L^2–scalar product $\langle \quad \rangle_\nu$ on $\mathscr{C}^\infty(X, L^p)$, the space of smooth sections of L^p, is given by

$$\langle \sigma_1, \sigma_2 \rangle_\nu := \int_X \langle \sigma_1(x), \sigma_2(x) \rangle_{L^p} \, d\nu(x) \,. \tag{3}$$

Let $P_{\nu,p}(x, x')$ $(x, x' \in X)$ be the smooth kernel of the orthogonal projection from $(\mathscr{C}^\infty(X, L^p), \langle \quad \rangle_\nu)$ onto $H^0(X, L^p)$, the space of the holomorphic sections of L^p on X, with respect to $d\nu(x')$. Note that $P_{\nu,p}(x, x') \in L^p_x \otimes L^{p*}_{x'}$. Following [3, §4], set

$$K_p(x, x') := |P_{\nu,p}(x, x')|^2_{h^{L^p}_x \otimes h^{L^{p*}}_{x'}}, \quad R_p := (\dim H^0(X, L^p))/\mathrm{Vol}(X, \nu), \tag{4}$$

here $\mathrm{Vol}(X, \nu) := \int_X d\nu$. Set $\mathrm{Vol}(X, dv_X) := \int_X dv_X$.

Let Q_{K_p} be the integral operator associated to K_p which is defined for $f \in \mathscr{C}^\infty(X)$,

$$Q_{K_p}(f)(x) := \frac{1}{R_p} \int_X K_p(x, y) f(y) d\nu(y). \tag{5}$$

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Let Δ be the (positive) Laplace operator on (X, g^{TX}) acting on the functions on X. We denote by $|\ \ |_{L^2}$ the L^2-norm on the function on X with respect to dv_X.

Theorem 1. *There exists a constant $C>0$ such that for any $f \in \mathscr{C}^\infty(X)$, $p \in \mathbb{N}$,*

$$\begin{aligned}
&\Big|\Big(Q_{K_p} - \frac{\mathrm{Vol}(X,\nu)}{\mathrm{Vol}(X, dv_X)}\eta \exp\Big(-\frac{\Delta}{4\pi p}\Big)\Big) f\Big|_{L^2} \leq \frac{C}{p}\,|f|_{L^2}\,,\\
&\Big|\Big(\frac{\Delta}{p}Q_{K_p} - \frac{\mathrm{Vol}(X,\nu)}{\mathrm{Vol}(X, dv_X)}\frac{\Delta}{p}\eta \exp\Big(-\frac{\Delta}{4\pi p}\Big)\Big) f\Big|_{L^2} \leq \frac{C}{p}\,|f|_{L^2}\,.
\end{aligned} \tag{6}$$

Moreover, (6) is uniform in that there is an integer s such that if all data h^L, $d\nu$ run over a set which are bounded in $\mathscr{C}^s$ and that g^{TX}, dv_X are bounded from below, then the constant C is independent of h^L, $d\nu$.

Proof. We explain at first the full asymptotic expansion of $P_{\nu,p}(x,x')$ from [2, Theorem 4.18′] and [4, §3.4]. For more details on our approach we also refer the readers to the recent book [5].

Let $E = \mathbb{C}$ be the trivial holomorphic line bundle on X. Let h^E the metric on E defined by $|1|^2_{h^E} = 1$, here 1 is the canonical unity element of E. We identify canonically L^p to $L^p \otimes E$ by Section 1.

As in [4, §3.4], let h^E_ω be the metric on E defined by $|1|^2_{h^E_\omega} = \eta^{-1}$, here 1 is the canonical unity element of E. Let $\langle\ \ \rangle_\omega$ be the Hermitian product on $\mathscr{C}^\infty(X, L^p \otimes E) = \mathscr{C}^\infty(X, L^p)$ induced by h^L, h^E_ω, dv_X as in (3). Then by (2),

$$(\mathscr{C}^\infty(X, L^p\otimes E), \langle\ \ \rangle_\omega) = (\mathscr{C}^\infty(X, L^p), \langle\ \ \rangle_\nu). \tag{7}$$

Observe that $H^0(X, L^p \otimes E)$ does not depend on g^{TX}, h^L or h^E. If $P_{\omega,p}(x,x')$, $(x, x' \in X)$ denotes the smooth kernel of the orthogonal projection $P_{\omega,p}$ from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle\cdot,\cdot\rangle_\omega)$ onto $H^0(X, L^p\otimes E) = H^0(X, L^p)$ with respect to $dv_X(x)$, from (2), as in [4, (3.38)], we have

$$P_{\nu,p}(x,x') = \eta(x')\, P_{\omega,p}(x,x'). \tag{8}$$

For $f \in \mathscr{C}^\infty(X)$, set

$$\begin{aligned}
K_{\omega,p}(x,x') &= |P_{\omega,p}(x,x')|^2_{(h^{L^p}\otimes h^E_\omega)_x \otimes (h^{L^{p*}}\otimes h^{E*}_\omega)_{x'}},\\
(K_{\omega,p}f)(x) &= \int_X K_{\omega,p}(x,y) f(y) dv_X(y).
\end{aligned} \tag{9}$$

By the definition of the metric h^E, h^E_ω, if we denote by 1^* the dual of the section 1 of E, we know

$$1 = |1\otimes 1^*|^2_{h^E\otimes h^{E*}}(x,x') = |1\otimes 1^*|^2_{h^E_\omega\otimes h^{E*}_\omega}(x,x')\eta(x)\eta^{-1}(x'). \tag{10}$$

Recall that we identified (L^p, h^{L^p}) to $(L^p\otimes E, h^{L^p}\otimes h^E)$ by Section 1. Thus from (4), (8) and (10), we get

$$K_p(x,x') = |P_{\nu,p}(x,x')|^2_{(h^{L^p}\otimes h^E)_x \otimes (h^{L^{p*}}\otimes h^{E*})_{x'}} = \eta(x)\,\eta(x')\,K_{\omega,p}(x,x'), \tag{11}$$

and from (2), (5) and (11),

$$Q_{K_p}(f)(x) = \frac{1}{R_p}\int_X K_{\omega,p}(x,y)\eta(x) f(y) dv_X(y). \tag{12}$$

Now for the kernel $P_{\omega,p}(x,x')$, we can apply the full asymptotic expansion [2, Theorem 4.18′]. In fact let $\overline{\partial}^{L^p\otimes E,*_\omega}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ on the Dolbeault complex $\Omega^{0,\bullet}(X,L^p\otimes E)$ with the scalar product induced by g^{TX}, h^L, h^E_ω, dv_X as in (3), and set

$$D_p=\sqrt{2}(\overline{\partial}^{L^p\otimes E}+\overline{\partial}^{L^p\otimes E,*_\omega}). \tag{13}$$

Then $H^0(X,L^p\otimes E)=\operatorname{Ker}D_p$ for p large enough, and D_p is a Dirac operator, as $g^{TX}(\cdot,\cdot)=\omega(\cdot,J\cdot)$ is a Kähler metric on TX.

Let ∇^E be the holomorphic Hermitian connection on (E,h^E_ω). Let ∇^{TX} be the Levi-Civita connection on (TX,g^{TX}). Let R^E, R^{TX} be the corresponding curvatures.

Let a^X be the injectivity radius of (X,g^{TX}). We fix $\varepsilon\in]0,a^X/4[$. We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε. We identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ by using the exponential map of (X,g^{TX}).

We fix $x_0\in X$. For $Z\in B^{T_{x_0}X}(0,\varepsilon)$ we identify (L_Z,h^L_Z), (E_Z,h^E_Z) and $(L^p\otimes E)_Z$ to $(L_{x_0},h^L_{x_0})$, $(E_{x_0},h^E_{x_0})$ and $(L^p\otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p\otimes E}$ along the curve $\gamma_Z:[0,1]\ni u\to\exp^X_{x_0}(uZ)$. Then under our identification, $P_{\omega,p}(Z,Z')$ is a function on $Z,Z'\in T_{x_0}X$, $|Z|,|Z'|\le\varepsilon$, we denote it by $P_{\omega,p,x_0}(Z,Z')$. Let $\pi:TX\times_X TX\to X$ be the natural projection from the fiberwise product of TX on X. Then we can view $P_{\omega,p,x_0}(Z,Z')$ as a smooth function on $TX\times_X TX$ (which is defined for $|Z|,|Z'|\le\varepsilon$) by identifying a section $S\in\mathscr{C}^\infty(TX\times_X TX,\pi^*\operatorname{End}(E))$ with the family $(S_x)_{x\in X}$, where $S_x=S|_{\pi^{-1}(x)}$, $\operatorname{End}(E)=\mathbb{C}$.

We choose $\{w_i\}_{i=1}^n$ an orthonormal basis of $T^{(1,0)}_{x_0}X$, then $e_{2j-1}=\frac{1}{\sqrt{2}}(w_j+\overline{w}_j)$ and $e_{2j}=\frac{\sqrt{-1}}{\sqrt{2}}(w_j-\overline{w}_j)\,,j=1,\dots,n$ forms an orthonormal basis of $T_{x_0}X$. We use the coordinates on $T_{x_0}X\simeq\mathbb{R}^{2n}$ where the identification is given by

$$(Z_1,\cdots,Z_{2n})\in\mathbb{R}^{2n}\longrightarrow\sum_i Z_ie_i\in T_{x_0}X. \tag{14}$$

In what follows we also introduce the complex coordinates $z=(z_1,\cdots,z_n)$ on $\mathbb{C}^n\simeq\mathbb{R}^{2n}$. By [2, (4.114)] (cf. [4, (1.91)]), set

$$P^N(Z,Z')=\exp\Big(-\frac{\pi}{2}\sum_i\big(|z_i|^2+|z'_i|^2-2z_i\overline{z}'_i\big)\Big). \tag{15}$$

Then P^N is the classical Bergman kernel on $\mathbb{C}^n$ (cf. [4, Remark 1.14]) and

$$|P^N(Z,Z')|^2=e^{-\pi|Z-Z'|^2}. \tag{16}$$

By [2, Proposition 4.1], for any $l,m\in\mathbb{N}$, $\varepsilon>0$, there exists $C_{l,m,\varepsilon}>0$ such that for $p\ge1$, $x,x'\in X$,

$$|P_{\omega,p}(x,x')|_{\mathscr{C}^m(X\times X)}\le C_{l,m,\varepsilon}\,p^{-l}\quad\text{if } d(x,x')\ge\varepsilon. \tag{17}$$

Here the $\mathscr{C}^m$-norm is induced by ∇^L, ∇^E, ∇^{TX} and h^L,h^E,g^{TX}.

By [2, Theorem 4.18′], there exist $J_r(Z,Z')$ polynomials in Z,Z', such that for any $k,m,m'\in\mathbb{N}$, there exist $N\in\mathbb{N},C>0,C_0>0$ such that for $\alpha,\alpha'\in\mathbb{N}^n$, $|\alpha|+|\alpha'|\le m$,

$Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \le \varepsilon$, $x_0 \in X$, $p \ge 1$,

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} P_{\omega,p,x_0}(Z,Z') - \sum_{r=0}^k (J_r P^N)(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} \right) \right|_{\mathscr{C}^{m'}(X)} \\ \le C p^{-(k+1-m)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0 \sqrt{p}|Z - Z'|) + \mathscr{O}(p^{-\infty}). \tag{18}$$

Here $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$ norm for the parameter $x_0 \in X$. The term $\mathscr{O}(p^{-\infty})$ means that for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1} p^{-l}$. (In fact, by [2, Theorems 4.6 and 4.17, (4.117)] (cf. [4, Theorem 1.18, (1.31)]), the polynomials $J_r(Z, Z')$ have the same parity as r and $\deg J_r(Z, Z') \le 3r$, whose coefficients are polynomials in R^{TX}, R^E and their derivatives of order $\leqslant r - 1$).

Now we claim that in (18),

$$J_0 = 1, \quad J_1(Z, Z') = 0. \tag{19}$$

In fact, let $dv_{T_{x_0}X}$ be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, and κ be the function defined by

$$dv_X(Z) = \kappa(x_0, Z) dv_{T_{x_0}X}(Z). \tag{20}$$

Then (also cf. [4, (1.31)])

$$\kappa(x_0, Z) = 1 + \frac{1}{6} \left\langle R^{TX}_{x_0}(Z, e_i)Z, e_i \right\rangle_{x_0} + \mathscr{O}(|Z|^3). \tag{21}$$

As we only work on $\mathscr{C}^\infty(X, L^p \otimes E)$, by [2, (4.115)], we get the first equation in (19).

Recall that in the normal coordinate, after the rescaling $Z \to Z/t$ with $t = \frac{1}{\sqrt{p}}$, we get an operator $\mathscr{L}_t$ from the restriction of D_p^2 on $\mathscr{C}^\infty(X, L^p \otimes E)$ which has the following formal expansion (cf. [2, (1.104)], [4, Theorem 1.4]),

$$\mathscr{L}_t = \mathscr{L} + \sum_{r=1}^\infty \mathscr{Q}_r t^r. \tag{22}$$

Now, from [2, Theorem 5.1] (or [4, (1.87), (1.97)]),

$$\mathscr{L} = \sum_{j=1}^n (-2\tfrac{\partial}{\partial z_i} + \pi \overline{z}_i)(2\tfrac{\partial}{\partial \overline{z}_i} + \pi z_i), \quad \mathscr{Q}_1 = 0. \tag{23}$$

(In fact, $P^N(Z, Z')$ is the smooth kernel of the orthogonal projection from $L^2(\mathbb{R})$ onto $\mathrm{Ker}(\mathscr{L})$). Thus from [2, (4.107)] (cf. [4, (1.111)]), (21) and (23) we get the second equation of (19).

Note that $|P_{\omega,p,x_0}(Z, Z')|^2 = P_{\omega,p,x_0}(Z, Z')\overline{P_{\omega,p,x_0}(Z, Z')}$, thus from (9), (18) and (19), there exist $J'_r(Z, Z')$ polynomials in Z, Z' such that

$$\left| \frac{1}{p^{2n+1}} \Delta_Z \Big(K_{\omega,p,x_0}(Z, Z') - \Big(1 + \sum_{r=2}^k p^{-r/2} J'_r(\sqrt{p}Z, \sqrt{p}Z')\Big) e^{-\pi p|Z-Z'|^2} \Big) \right| \\ \le C p^{-(k+1)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C_0 \sqrt{p}|Z - Z'|) + \mathscr{O}(p^{-\infty}). \tag{24}$$

For a function $f \in \mathscr{C}^\infty(X)$, we denote it as $f(x_0, Z)$ a family (with parameter x_0) of function on Z in the normal coordinate near x_0. Now, for any polynomial $Q_{x_0}(Z')$, we

define the operator

$$(\mathscr{Q}_p f)(x_0) = p^n \int_{|Z'|\leq \varepsilon} Q_{x_0}(\sqrt{p}Z') e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z'). \tag{25}$$

Then we observe that there exists $C_1 > 0$ such that for any $p \in \mathbb{N}$, $f \in \mathscr{C}^\infty(X)$, we have

$$|\mathscr{Q}_p f|_{L^2} \leq C_1 |f|_{L^2}. \tag{26}$$

In fact,

$$\begin{aligned} |\mathscr{Q}_p f|^2_{L^2} \leq & \int_X dv_X(x_0) \Big\{ p^n \Big(\int_{|Z'|\leq\varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} dv_X(x_0, Z') \Big) \\ & \times p^n \Big(\int_{|Z'|\leq\varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} |f(x_0, Z')|^2 dv_X(x_0, Z') \Big) \Big\} \\ & \leq C' \int_X dv_X(x_0) p^n \int_{|Z'|\leq\varepsilon} |Q_{x_0}(\sqrt{p}Z')| e^{-\pi p|Z'|^2} |f(x_0, Z')|^2 dv_X(x_0, Z') \\ & \leq C_1 |f|^2_{L^2}. \end{aligned} \tag{27}$$

Observe that in the normal coordinate, at $Z = 0$, $\Delta_Z = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial Z_j^2}$. Thus

$$(\Delta_Z e^{-\pi p|Z-Z'|^2})|_{Z=0} = 4\pi p(n - \pi p|Z'|^2) e^{-\pi p|Z'|^2}. \tag{28}$$

Thus from (16), (18), (19), (24) and (26), we get

$$\begin{aligned} & \left| p^{-n} K_{\omega,p} f - p^n \int_{|Z'|\leq\varepsilon} e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\ & \left| p^{-n-1} \Delta K_{\omega,p} f - 4\pi p^n \int_{|Z'|\leq\varepsilon} (n - \pi p|Z'|^2) e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}. \end{aligned} \tag{29}$$

Set

$$\begin{aligned} & K_{\eta,\omega,p}(x, y) = \langle d\eta(x), d_x K_{\omega,p}(x, y) \rangle_{g^{T^*X}}, \\ & (K_{\eta,\omega,p} f)(x) = \int_X K_{\eta,\omega,p}(x, y) f(y) dv_X(y). \end{aligned} \tag{30}$$

Then from (18), (19) and (26), we get

$$\left| p^{-n-1} K_{\eta,\omega,p} f - 2\pi p^n \int_{|Z'|\leq\varepsilon} \sum_{i=1}^{2n} (\frac{\partial}{\partial Z_i}\eta)(x_0, 0) Z_i' e^{-\pi p|Z'|^2} f(x_0, Z') dv_X(x_0, Z') \right|_{L^2} \leq \frac{C}{p} |f|_{L^2}. \tag{31}$$

Let $e^{-u\Delta}(x, x')$ be the smooth kernel of the heat operator $e^{-u\Delta}$ with respect to $dv_X(x')$. Let $d(x, y)$ be the Riemannian distance from x to y on (X, g^{TX}). By the heat kernel expansion in [1, Theorems 2.23, 2.26], there exist $\Phi_i(x, y)$ smooth functions on $X \times X$ such that when $u \to 0$, we have the following asymptotic expansion

$$\left| \frac{\partial^l}{\partial u^l} \Big(e^{-u\Delta}(x, y) - (4\pi u)^{-n} \sum_{i=0}^k u^i \Phi_i(x, y) e^{-\frac{1}{4u} d(x,y)^2} \Big) \right|_{\mathscr{C}^m(X\times X)} = \mathscr{O}(u^{k-n-l-\frac{m}{2}+1}), \tag{32}$$

and

$$\Phi_0(x,y) = 1. \tag{33}$$

If we still use the normal coordinate, then by (32), there exist $\phi_{i,x_0}(Z') := \Phi_i(0,Z')$ such that uniformly for $x_0 \in X$, $Z' \in T_{x_0}X, |Z'| \leq \varepsilon$, we have the following asymptotic expansion when $u \to 0$,

$$\left| \frac{\partial^l}{\partial u^l} \Big(e^{-u\Delta}(0,Z') - (4\pi u)^{-n} \Big(1 + \sum_{i=1}^k u^i \phi_{i,x_0}(Z') \Big) e^{-\frac{1}{4u}|Z'|^2} \Big) \right| = \mathscr{O}(u^{k-n-l+1}), \tag{34}$$

and

$$\begin{aligned} \Big| \langle d\eta(x_0), d_{x_0} e^{-u\Delta} \rangle_{g^{T^*X}}(0,Z') \\ - (4\pi u)^{-n} \sum_{i=1}^{2n} (\frac{\partial}{\partial Z_i}\eta)(x_0,0) \frac{Z'_i}{2u} \Big(1 + \sum_{i=1}^k u^i \phi_{i,x_0}(Z') \Big) \Big) e^{-\frac{1}{4u}|Z'|^2} \\ - (4\pi u)^{-n} \sum_{i=1}^k u^i \langle d\eta(x_0), (d_{x_0}\Phi_i)(0,Z') \rangle e^{-\frac{1}{4u}|Z'|^2} \Big| = \mathscr{O}(u^{k-n+\frac{1}{2}}). \end{aligned} \tag{35}$$

Observe that

$$\frac{1}{p} \Delta \exp\Big(- \frac{\Delta}{4\pi p} \Big) = -\frac{1}{p} (\tfrac{\partial}{\partial u} e^{-u\Delta})|_{u=\frac{1}{4\pi p}}. \tag{36}$$

Now from (26), (29)–(36), we get

$$\begin{aligned} \Big| \Big(p^{-n} K_{\omega,p} - \exp\Big(- \frac{\Delta}{4\pi p} \Big) \Big) f \Big|_{L^2} \leq \frac{C}{p} |f|_{L^2}, \\ \Big| \frac{1}{p} \Big(p^{-n} \Delta K_{\omega,p} - \Delta \exp\Big(- \frac{\Delta}{4\pi p} \Big) \Big) f \Big|_{L^2} \leq \frac{C}{p} |f|_{L^2}. \end{aligned} \tag{37}$$

and

$$\Big| \frac{1}{p} \Big(p^{-n} K_{\eta,\omega,p} - \langle d\eta, d\exp(-\frac{\Delta}{4\pi p}) \rangle \Big) f \Big|_{L^2} \leq \frac{C}{p} |f|_{L^2}. \tag{38}$$

Note that

$$\begin{aligned} (\Delta \eta K_{\omega,p})(x,y) = (\Delta\eta)(x) K_{\omega,p}(x,y) + \eta(x) \Delta_x K_{\omega,p}(x,y) \\ - 2 \langle d\eta(x), d_x K_{\omega,p}(x,x') \rangle_{g^{T^*X}}, \end{aligned} \tag{39}$$

and $R_p = \frac{\mathrm{Vol}(X,dv_X)}{\mathrm{Vol}(X,\nu)} p^n + \mathscr{O}(p^{n-1})$. From (12), (37)-(39), we get (6).

To get the last part of Theorem 1, as we noticed in [2, §4.5], the constants in (18) will be uniformly bounded under our condition, thus we can take C in (6), (37)and (38) independent of h^L, $d\nu$. □

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